JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 27, No. 6, November–December 2004

Nonlinear Control Allocation Using Piecewise Linear Functions

Michael A. Bolender* and David B. Doman†
U.S. Air Force Research Laboratory, Wright–Patterson Air Force Base, Ohio 45433

A novel method is presented for the solution of the control allocation problem where the control variable rates or moments are nonlinear functions of control position. Historically, control allocation has been performed under the assumption that a linear relationship exists between the control induced moments and the control effector displacements. However, aerodynamic databases are discrete valued and almost always stored in multidimensional lookup tables, where it is assumed that the data are connected by piecewise linear functions. The approach that is presented utilizes this piecewise linear assumption for the control effector moment data. This assumption allows the control allocation problem to be cast as a piecewise linear program that can account for nonlinearities in the moment/effector relationships, as well as to enforce position constraints on the effectors. The piecewise linear program is then recast as a mixed-integer linear program. It is shown that this formulation accurately solves the control allocation problem when compared to the aerodynamic model. It is shown that the control effector commands for a reentry vehicle by the use of the piecewise linear control allocation method are markedly improved when compared to the performance of more traditional control allocation approaches that use a linear relationship between the control moments and the effectors. The technique is also applied to determine those flight conditions (angle of attack and Mach number) at which the reentry vehicle can be trimmed for the purpose of providing constraint estimates to trajectory reshaping algorithms.

Introduction

THE utilization of reconfigurable control laws for autonomous vehicles has resulted in an increased interest in the subject of control allocation. Reconfigurable control laws require a control allocation algorithm when the number of control effectors exceeds the number of controlled variables. Typically, on reusable launch vehicles, there are only a few controlled variables and some minimal set of effectors to satisfy redundancy requirements. When the number of effectors exceeds the number of controlled variables, it is quite common that the desired commands can be achieved in many different ways. A control allocation algorithm can be used to find a set of control effector positions that meet some desired objective in addition to delivering the desired moments. Additionally, the control effectors are subject to position and/or rate limiting constraints that can be enforced by a well-designed control allocation algorithm.

In situations where an aircraft has experienced one or more control effector failures, the control allocator can be used to reconfigure the remaining effectors to satisfy the control objective. If it is not physically possible to satisfy a control objective exactly, then control allocation can be used to minimize the extent to which the objective is not satisfied. To accomplish this, the control allocator must have an accurate estimate of the control effector's moment producing capabilities. Control allocation has historically been performed under the assumption that a linear relationship exists between the control induced moments and the control effector displacements, despite the fact that the forces and moments produced by aircraft control surfaces are almost always nonlinear functions of control surface displacement. This assumption of linearity is usually sufficient for flight when all control effectors remain operable because most effectors are approximately linear over some range of deflection in at least one axis. However, if an effector failure forces the unfailed control effectors to operate in a highly nonlinear region of the control moment curve, the linear approximation may not be sufficiently accurate for the vehicle to be safely recovered. In this paper, we will refer to any control allocation approach that assumes the moments are linear functions of the control effectors as linear control allocation. If the control moments are assumed to be nonlinear functions of the effectors, this will be referred to as nonlinear control allocation. Note that a class of what we refer to as linear control allocation approaches can accommodate nonlinearities such as rate and position constraints. Here, we are developing a technique that can enforce such constraints in addition to making use of known nonlinear relationships between the moments and the effector positions.

Control allocation is vital to the adaptive/reconfigurable flight control systems that are now being developed for both autonomous and manned aircraft. These control systems are gaining favor due to their robustness properties, especially when an aircraft experiences control effector failures. Several examples of dynamic inversion-based adaptive/reconfigurable flight control systems can be found in the literature.[1–3] Control allocation algorithms also play an important part in the online determination of an accurate footprint for a reusable launch vehicle that has experienced a control effector failure.[4] Buffington[5] has demonstrated the application of a dynamic inversion control law, along with a control allocation algorithm to a tailless fighter application. Tailless aircraft have reduced directional stability due to the lack of a vertical tail and rudder for directional control. Ailerons or spoilers are examples of conventional control surfaces that can be used to provide directional control; however, these control effectors lack the control authority of a typical rudder, requiring that a mix of control effectors be used to generate the appropriate moments. Furthermore, left/right pairs of effectors such as ailerons and elevators typically have highly nonlinear contributions to the yawing moment. This is especially true at low angles of attack where the effects of parasitic drag dominate induced drag. Nonlinear control allocation is required to make beneficial use of the nonlinear effects that are often rejected as disturbances by a control system that uses a linear control allocator.

Comprehensive surveys of existing control allocation techniques have been presented by Bodson[6] as well as Page and Steinberg.[7,8] Page and Steinberg[7,8] have performed extensive simulation studies and documented the open- and closed-loop performance of the most common linear control allocation algorithms. On the other hand, Bodson[6] has compared constrained, numerical-based optimization methods for control allocation to determine their feasibility for use in a real-time flight control system.

Received 30 January 2003; presented as Paper 2003-5357 at the 2003 AIAA Guidance, Navigation, Control Conference, Austin, TX, 11–14 August 2003; revision received 25 March 2004; accepted for publication 8 March 2004. This material is declared a work of the U.S. Government and is not subject to copyright protection in the United States. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0731-5090/04 $10.00 in correspondence with the CCC.

*Visiting Scientist, Control Theory and Optimization Branch. Senior Member AIAA.

†Senior Aerospace Engineer, Control Theory and Optimization Branch. Senior Member AIAA.

Numerous linear control allocation algorithms[5,9–11] are currently available and are well documented in the literature. Buffington's[5] approach was to solve to the control allocation problem using a two branch approach. The first branch of this control allocation approach is a check for control deficiency. The control deficiency branch determines whether the control effectors are capable of producing a given moment command without violating control effector limits. Control deficiency is checked by the solution of an optimization problem that minimizes the ℓ_1 norm of the error between the desired moment and the moments produced by control effectors. The value of the ℓ_1 norm indicates the feasibility of the commanded moment. If the commanded moment is infeasible, the solution of the control deficiency branch is the solution to the control allocation problem. When the commanded moment is determined to be feasible, that is, the ℓ_1 norm is identically zero, then there exists at least one feasible solution to the problem, and a second optimization problem is solved. This second branch of the control allocation problem is the control sufficiency branch, and it is used to provide solution uniqueness by minimizing the control effector positions with respect to some preferred position in an ℓ_1 norm sense. The optimization problems in the multibranch approach can be converted into linear programming problems for the case where the control moments are linear functions of control effector displacements.

Recently, there have been several papers in the literature that have discussed relaxing the assumption that the control moments are linear functions of the control effectors. Recently, Doman and Oppenheimer[12] have implemented a control allocator that uses a linear approximation of the local slope of the control moment curve with an added intercept term to account more accurately for the nonlinear behavior of aerodynamic control effectors. The advantage of this approach is that the control allocation problem can still be posed as a linear program and solved efficiently with off-the-shelf software. The end result is that the accuracy is improved over the more traditional linear programming approaches to control allocation without adding additional complexity to the simplex algorithm. The downside of this approach is that the control moments are required to be monotonic functions of the effector displacement. Otherwise, modeling errors will occur that will result in incorrect control effector positions.

Doman and Sparks[13] devised a method that determines the nonlinear attainable moment set (AMS) for the two-moment case. More recently, Bolender and Doman[14] extended this work to three dimensions for the computation of the nonlinear AMS volume when the control moments about the third control axis were linear functions of effector position. At the present time, control allocation methods that utilize the knowledge that control moments are nonlinear functions of control effector displacement do not lend themselves to be applied in a real-time control system.

The objective of this paper is to demonstrate that the control allocation problem can be posed in a manner such that the moments that result from the corresponding control surface deflections are exactly the moments that are returned from the aerodynamic database. Aerodynamic data are typically discrete valued and stored in large, multidimensional arrays that are functions of flight condition, namely, angle of attack, sideslip angle, Mach number, and control surface deflection. For flight control system design and handling qualities analysis, it is commonplace to assume that the data are connected by piecewise linear functions. The method that we are proposing likewise assumes that the control moments are piecewise linear functions of control surface deflection and flight condition. We will then pose the control allocation problem as a piecewise linear program. The piecewise linear program will account for the nonlinearities of the aerodynamic data, and the solution to this problem will produce deflections that result in moments that match those found through linear interpolation of the aerodynamic data. This approach results in improved command tracking performance when compared to the linear control allocation approaches just discussed. The accuracy of the linear and piecewise linear control allocation approaches is compared when the two methods are interchanged in a dynamic inversion control law framework for a reentry vehicle with redundant control effectors. Also, we will show how this method can be applied to a flight envelope constraint estimation problem whose solution supports the computation of trajectories under failure conditions.

Dynamic Inversion Flight Control

Dynamic inversion controllers attempt to cancel and replace the dynamics of the plant being controlled with a set of desired dynamics. If the fidelity of the onboard reference model is high enough, then the dynamic inversion control law results in a closed-loop system that behaves like a decoupled bank of integrators. In the context of flight control, a common objective of a dynamic inversion control law is to provide good body-axis angular rate tracking.

It is assumed that a pilot or an outer-loop guidance system generates body-axis angular velocity commands P_c, Q_c, and R_c. The inner-loop dynamic inversion control law is designed such that the aircraft tracks these angular velocity commands (Fig. 1). The rotational dynamics for an aircraft can be written as

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{G}_B - \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} \tag{1}$$

where $\boldsymbol{\omega} = [P \;\; Q \;\; R]^T$, $\boldsymbol{I}$ is the moment-of-inertia tensor, and $\boldsymbol{G}_B$ are the moments acting on the vehicle. We can express $\boldsymbol{G}_B$ as a sum that includes moments that are due to the wing/body aerodynamics and propulsion system, which we will collectively refer to as the base moments and moments due to the control effectors:

$$\boldsymbol{G}_B = \boldsymbol{G}_{\text{base}}(\boldsymbol{\omega}, \boldsymbol{P}) + \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \begin{bmatrix} L \\ M \\ N \end{bmatrix}_{\text{base}} + \begin{bmatrix} L(\boldsymbol{\delta}) \\ M(\boldsymbol{\delta}) \\ N(\boldsymbol{\delta}) \end{bmatrix} \tag{2}$$

where $\boldsymbol{G}_{\text{base}}(\boldsymbol{\omega}, \boldsymbol{P})$ is the moment generated by the base engine/aerodynamic system and $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$ is the sum of the moments produced by the control effectors. The parameter vector $\boldsymbol{P}$ denotes measurable or estimable quantities that influence the body angular accelerations and includes variables such as Mach number, angle of attack, sideslip angle, and vehicle mass properties such as moments of inertia. Thus, we define

$$\boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{P}) \stackrel{\Delta}{=} \boldsymbol{G}_{\text{base}}(\boldsymbol{\omega}, \boldsymbol{P}) - \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} \tag{3}$$

The model used for the design of the dynamic inversion control law then becomes

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{P}) + \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) \tag{4}$$

and our objective is to find a control law that provides direct control over $\dot{\boldsymbol{\omega}}$ such that $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_{\text{des}}$, that is,

$$\boldsymbol{I}\dot{\boldsymbol{\omega}}_{\text{des}} = \boldsymbol{f}(\boldsymbol{\omega}, P) + \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) \tag{5}$$

therefore, the inverse control must satisfy

$$\boldsymbol{I}\dot{\boldsymbol{\omega}}_{\text{des}} - \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{P}) = \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) \tag{6}$$

Because there are more control effectors than controlled variables, a control allocation algorithm must be used to obtain a solution. Solution of this control allocation problem will be discussed in detail in a later section. Equation (6) states that the control effectors are to be used to correct for the difference between the desired accelerations and the accelerations due only to the base moments.

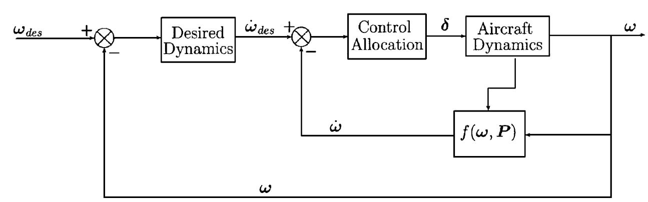


Fig. 1 Block diagram of inner-loop dynamic inversion control law.

Piecewise Linear Programming

Piecewise linear programming is an optimization method that allows one to approximate nonlinear programming problems that comprise separable functions. To solve the resulting approximation problem, a linear program can be formulated and then solved by the use of a modified simplex method.[15] A second option is to formulate the nonlinear program as a mixed-integer linear program.[16]

In terms of control allocation, the restriction of the approximation of separable functions by piecewise linear functions may appear to be overly restrictive. For most aircraft, the control-induced moments can be considered as separable because, in many cases, there are no significant aerodynamic interactions among the control effectors. In some instances, the cross-coupling of control effectors cannot be neglected, such as when control effectors are located downstream of other surfaces.

For purposes of illustration, we will approximate a single-valued function $f(x)$ by its piecewise linear approximation and show how to formulate the minimization of $f(x), x \in [a, b]$ as a piecewise linear program. The approach given hereafter for a single variable function can be generalized for multivariable, separable functions rather easily. Furthermore, we are not restricted to approximating only the objective function by a piecewise linear approximation because it is also possible to consider piecewise linear approximations of the constraints, if they are separable, within the same framework. A detailed discussion can be found in Ref. 15.

Without loss of generality, we begin by considering a function $f(x)$ of a single variable, defined on an interval $[a, b]$. Begin by the definition of a grid of K points spaced on the interval $[a, b]$ and denote these points as $x^{(k)}$, $k = 1, \ldots, K$ where $a = x^{(1)} < x^{(2)} < \cdots < x^{(k)} < \cdots < x^{(K)} = b$. Note that we are not restricted to a uniform spacing of the $x^{(k)}$. Furthermore, let $f^{(k)}$ denote the value of $f(x^{(k)})$. A piecewise linear approximation of $f(x)$ can then be constructed by the connection of $(x^{(k)}, f^{(k)})$ and $(x^{(k+1)}, f^{(k+1)})$ with a straight line, as shown in Fig. 2. The equation of the line connecting the points $(x^{(k)}, f^{(k)})$ and $(x^{(k+1)}, f^{(k+1)})$ is given by

$$\tilde{f}(x) = f^{(k)} + \frac{f^{(k+1)} - f^{(k)}}{x^{(k+1)} - x^{(k)}}\left(x - x^{(k)}\right) \tag{7}$$

where $x \in [x^{(k)}, x^{(k+1)}]$. There will be $K - 1$ such equations, one for each subinterval. Observe that on a given subinterval, x can be written as

$$x = \lambda^{(k)} x^{(k)} + \lambda^{(k+1)} x^{(k+1)} \tag{8}$$

where $\lambda^{(k)} \geq 0$ and $\lambda^{(k+1)} \geq 0$. The $\lambda^{(k)}$ are normalized such that

$$\lambda^{(k)} + \lambda^{(k+1)} = 1 \tag{9}$$

It can then be shown that Eq. (7) can be written as

$$\tilde{f}(x) = \lambda^{(k)} f^{(k)} + \lambda^{(k+1)} f^{(k+1)} \tag{10}$$

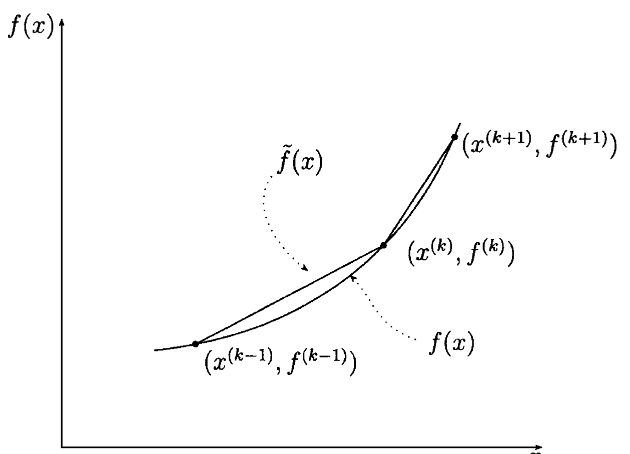


Fig. 2 Piecewise linear function approximation.

Therefore, in the interval $[x^{(1)}, x^{(K)}]$, each x and the approximate value $\tilde{f}(x)$ can be determined by assignment of the appropriate values to $\lambda^{(k)}$ and $\lambda^{(k+1)}$ that correspond to the subinterval in which x lies. Because x can only be defined on a single subinterval, all of the $\lambda^{(k)}$ that are not associated with that particular interval must all be equal to zero. As a result, we can express Eqs. (8) and (10) as

$$x = \sum_{k=1}^{K} \lambda^{(k)} x^{(k)} \tag{11}$$

$$\tilde{f}(x) = \sum_{k=1}^{K} \lambda^{(k)} f^{(k)} \tag{12}$$

subject to the following conditions:

$$\sum_{k=1}^{K} \lambda^{(k)} = 1 \tag{13}$$

$$\lambda^{(k)} \geq 0, \qquad k = 1, \ldots, K \tag{14}$$

$$\lambda^{(i)} \lambda^{(j)} = 0 \qquad \text{if} \qquad j > i + 1, \qquad i = 1, \ldots, K - 1 \tag{15}$$

Equation (15) is necessary to ensure that only points lying on piecewise linear segments that connect adjacent breakpoints are considered part of the approximating function. For example, given a value of x, no more than two of the $\lambda^{(k)}$ are allowed to be positive, and the two $\lambda^{(k)}$ also must be adjacent. If we consider a value of x where $\lambda^{(3)}$ and $\lambda^{(4)}$ are positive, with $\lambda^{(1)} = \lambda^{(2)} = 0$ and $\lambda^{(k)} = 0$, $k = 5, \ldots, K$, then the value of $\tilde{f}(x)$ lies on the approximating function between $x^{(3)}$ and $x^{(4)}$. On the other hand, if $\lambda^{(4)} > 0$ was to be replaced by $\lambda^{(6)} > 0$, and all other $\lambda^{(k)} = 0$, then the line connecting $x^{(3)}$ and $x^{(6)}$ would not be part of the approximating function. Furthermore, if we chose a value of x such that $x = x^{(k)}$ and $\tilde{f}(x) = f(x)$, then, from Eq. (13), $\lambda^{(k)} = 1$ and all other values of $\lambda = 0$. Last, note that one can always obtain a more accurate approximation of $f(x)$ by increasing the number of gridpoints; however, this obviously increases the size of problem.

Given that we now have a piecewise linear approximation to $f(x)$ and the additional constraints that result from the transformation, we are able to state the piecewise linear program that corresponds to the minimization of $f(x)$ on the interval $a \leq x \leq b$:

$$\min \tilde{f}(x) = \sum_{k=1}^{K} \lambda^{(k)} f^{(k)} \tag{16}$$

subject to

$$\sum_{k=1}^{K} \lambda^{(k)} = 1 \tag{17}$$

$$\lambda^{(k)} \geq 0 \tag{18}$$

Once the solution to the piecewise linear program is obtained, one uses Eq. (11) to find the corresponding value of x that gives an approximate minimum to $f(x)$. Finding a solution to a piecewise linear program requires an approach that ensures that Eq. (15) is satisfied.

Recall that Eq. (15) requires that no more than two adjacent $\lambda^{(k)}$ are allowed to be nonzero. Therefore, to find an optimal feasible solution to the piecewise linear program, one of two approaches must be taken. One approach is to solve the problem with the simplex method with a restricted basis entry rule.[15] A second approach is to formulate Eq. (15) with binary decision variables[16] that will constrain x to be on only one subinterval. The result will be yet another increase in the size of the problem beyond what was necessary for the piecewise linear approximation. The addition of the binary variables transforms the piecewise linear programming problem into a mixed-integer linear program (MILP). We will take the latter approach because it is sufficient for demonstrating the validity of the approach and also because of the availability of an open-source code (GNU Linear Programming Kit) that solves linear programs and mixed-integer linear programs.

Transformation of the Piecewise Linear Program to a MILP

Begin by considering the piecewise linear approximation shown in Fig. 2. Note that if there are K breakpoints, then there are $K-1$ linear segments. We assign a variable $y^{(k)}$ that corresponds to the kth linear segment of the piecewise linear approximation such that

$$y^{(k)} = \begin{cases} 1 & \text{if} \quad \lambda^{(k)} \neq 0 \quad \text{and} \quad \lambda^{(k+1)} \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

for $k = 1, \dots, K-1$. Next, we make the observation that if $\lambda^{(1)} \neq 0$ and $\lambda^{(2)} \neq 0$, then

$$\lambda^{(1)} \leq y^{(1)} \tag{20}$$

$$\lambda^{(2)} \leq y^{(1)} \tag{21}$$

where $y^{(1)} = 1$. However, if we are on the segment where $\lambda^{(2)} \neq 0$ and $\lambda^{(3)} \neq 0$, such that $y^{(2)} = 1$, then

$$\lambda^{(2)} \leq y^{(2)} \tag{22}$$

$$\lambda^{(3)} \leq y^{(2)} \tag{23}$$

If we proceed in this manner, we observe that the following restrictions can be placed on the $\lambda^{(k)}$

$$\lambda^{(1)} \leq y^{(1)} \tag{24}$$

$$\lambda^{(k)} \leq y^{(k-1)} + y^{(k)}, \qquad k = 2, \dots, K-1 \tag{25}$$

$$\lambda^{(K)} \leq y^{(K-1)} \tag{26}$$

The rationale behind Eq. (25) is as follows: The $\lambda^{(k)}$ that correspond to points that are interior to the interval, that is, they are not the endpoints of the interval on which x is defined, can be associated with one of two line segments. A particular $\lambda^{(k)}$ is the endpoint for the line segment immediately preceding it, in addition to the line segment that comes immediately after it. Only one of these two line segments may be active at any time; therefore, the right-hand side of Eq. (25) is never greater than one. In addition to Eqs. (24–26), we have an additional constraint to ensure that only one of the $K-1$ line segments is active; hence, only one of the $y^{(k)}$ can be equal to one:

$$\sum_{k=1}^{K-1} y^{(k)} = 1 \tag{27}$$

By including Eqs. (24–27) into the piecewise linear program, we transform it into a MILP. The transformed optimization problem is stated as follows:

$$\min \tilde{f}(x) = \sum_{k=1}^{K} \lambda^{(k)} f^{(k)} \tag{28}$$

subject to

$$\sum_{k=1}^{K} \lambda^{(k)} = 1 \tag{29}$$

$$\lambda^{(k)} \geq 0 \tag{30}$$

$$\lambda^{(1)} \leq y^{(1)} \tag{31}$$

$$\lambda^{(k)} \leq y^{(k-1)} + y^{(k)}, \qquad k = 2, \dots, K-1 \tag{32}$$

$$\lambda^{(K)} \leq y^{(K-1)} \tag{33}$$

$$\sum_{k=1}^{K-1} y^{(k)} = 1 \tag{34}$$

$$y^{(k)} \in \{0, 1\} \tag{35}$$

By including the additional constraints that are necessary to complete the transformation of the piecewise linear program, we have added an additional $K-1$ decision variables to the problem. This does not include any slack or surplus variables that may be required by the solver. The slack and surplus variables will further increase the number of decision variables. The solution to the MILP is obtained with a branch-and-bound algorithm. Technical details on the branch-and-bound algorithm can be found in Ref. 16.

Formulation of Control Allocation Problems as MILP

For typical aircraft, there are three controlled variables (moments) and three control surfaces, resulting in a square system of equations that form a unique mapping of the control moments to the control surfaces. On the other hand, aircraft such as the X-33, X-40A, F/A-18 HARV, F-15 ACTIVE, and AFTI/F-16 have more control surfaces than controlled variables. The resulting underdetermined system requires that a control allocation algorithm be used to ensure that Eq. (6) is satisfied. There are often an infinite number of solutions for given values of the controlled variables; therefore, control allocation is often cast as an optimization problem to obtain a solution with some desired properties. Such objectives commonly include the minimization of control effector displacement or the minimization of the control moment error. The control allocation formulation that is used in this paper follows the multibranch approach, similar to that posed by Buffington[5]; however, the assumption that the control induced moments are linear functions of the control displacements has been removed.

In this section, the control allocation problems are formulated as piecewise linear programs. Recall that piecewise linear programming is a method for approximating a nonlinear programming problem that comprises separable functions. The resulting piecewise linear program approach minimizes a performance index that includes the linear terms of the original nonlinear program and/or the piecewise linear approximation of any nonlinear functions subject to linear and piecewise linear constraints. We will begin with a discussion of the linear control allocation problem and the equivalent linear programming problem. We will then formulate the piecewise linear programs for the control deficiency and control sufficiency branches. The piecewise linear programs are then subsequently transformed and solved as MILP by the use of a branch-and-bound algorithm. The resulting mixed-integer program is much more complex and difficult to solve than a linear program. The difference is that we are now able to achieve more accurately the moments that we desire for any given feasible moment command.

Control Deficiency Branch

For this discussion, let us begin by examining the case where the moments are approximated as linear functions of control positions subject to instantaneous position constraints. The control deficiency branch is used to test whether there exists a set of control effector positions that will satisfy Eq. (6). For convenience, we will refer to the left-hand side of Eq. (6) as $\boldsymbol{d}_{\text{des}}$:

$$\boldsymbol{d}_{\text{des}} \triangleq \boldsymbol{I}\dot{\boldsymbol{\omega}}_{\text{des}} - \boldsymbol{f}(\boldsymbol{\omega}, \boldsymbol{P}) = \boldsymbol{G}_{\delta}(\boldsymbol{P})\boldsymbol{\delta} \triangleq \boldsymbol{B}\boldsymbol{\delta} \tag{36}$$

where $\boldsymbol{G}_{\delta}(\boldsymbol{P})\boldsymbol{\delta} \approx \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$. Then $\boldsymbol{G}_{\delta}(\boldsymbol{P})$ is a $3 \times m$ matrix where $G_{\delta_{ij}} = B_{ij} = \partial G_i / \partial \delta_j$. If it is not feasible to obtain $\boldsymbol{d}_{\text{des}} = \boldsymbol{B}\boldsymbol{\delta}$ due to control effector constraints, then the difference between the desired and actual effector-induced body-axis accelerations is minimized. The control deficiency branch is stated in terms of the minimization of a weighted one-norm performance index:

$$\min_{\delta} J_D = \left\| \boldsymbol{w}_a^T (\boldsymbol{B}\boldsymbol{\delta} - \boldsymbol{d}_{\text{des}}) \right\|_1 \tag{37}$$

subject to

$$\underline{\boldsymbol{\delta}} \leq \boldsymbol{\delta} \leq \bar{\boldsymbol{\delta}} \tag{38}$$

The column vector $\boldsymbol{w}_a$ has positive elements and is used to prioritize the axes. The bounds on the control effectors are defined to be $\underline{\boldsymbol{\delta}}$ and $\bar{\boldsymbol{\delta}}$, where

$$\bar{\boldsymbol{\delta}} = \min(\boldsymbol{\delta}_u, \Delta T \dot{\boldsymbol{\delta}}_{\max} + \boldsymbol{\delta}), \qquad \underline{\boldsymbol{\delta}} = \max(\boldsymbol{\delta}_l, -\Delta T \dot{\boldsymbol{\delta}}_{\max} + \boldsymbol{\delta}) \tag{39}$$

where $\boldsymbol{\delta}_u$ is the upper position limit vector, $\boldsymbol{\delta}_l$ is the lower position limit vector, $\dot{\boldsymbol{\delta}}_{\max}$ is a vector of the effector rate limits, and ΔT is the inner-loop flight control system update rate. The optimization problem posed in Eq. (38) may be transformed into the following linear programming problem[5]:

$$\min_{\delta_s} J_D = \begin{bmatrix} 0 & \cdots & 0 & \boldsymbol{w}_a^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\delta}_s \end{bmatrix} \tag{40}$$

subject to

$$\begin{bmatrix} \boldsymbol{\delta}_s \\ -\boldsymbol{\delta} \\ \boldsymbol{\delta} \\ -\boldsymbol{B\delta} + \boldsymbol{\delta}_s \\ \boldsymbol{B\delta} + \boldsymbol{\delta}_s \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{0} \\ -\bar{\boldsymbol{\delta}} \\ \underline{\boldsymbol{\delta}} \\ -\boldsymbol{d}_{\mathrm{des}} \\ \boldsymbol{d}_{\mathrm{des}} \end{bmatrix} \tag{41}$$

where $\boldsymbol{\delta}_s$ has the same dimension as the set of controlled variables. If $J_D = 0$, then the commanded controlled variable rates are achievable, and excess control power may be available to optimize subobjectives. If $J_D \neq 0$, the commanded controlled variable rates are not achievable, and the control allocation algorithm provides a vector of effector commands that minimize the deficiency.

Control Deficiency Branch as a MILP

To transform the ℓ_1 optimization of the control deficiency branch to a piecewise linear program, we will focus on the transformed linear program as defined in Eqs. (40) and (41). The transformation of the control allocation problem to the piecewise linear program will involve the control effectors $\delta_i, i = 1, \ldots, m$, and the terms containing $\boldsymbol{B\delta}$. We want to replace $\boldsymbol{B\delta}$, where an element in the ith row of $\boldsymbol{B}$ is a linear approximation of the control moment produced by δ_i, by a piecewise linear approximation of the control moments as a function of control effector position.

Let $L_i(\delta_i)$, $M_i(\delta_i)$, and $N_i(\delta_i)$ denote the rolling, pitching, and yawing moments produced by deflection of the ith control surface δ_i. The piecewise linear approximation of $L_i(\delta_i)$ can be written as

$$L_i(\delta_i) = \sum_{k=1}^{K_i} L_i^{(k)} \lambda_i^{(k)} \tag{42}$$

$$= \begin{bmatrix} L_i^{(1)} L_i^{(2)} \ldots L_i^{(K_i)} \end{bmatrix} \begin{bmatrix} \lambda_i^{(1)} \\ \lambda_i^{(2)} \\ \vdots \\ \lambda_i^{(K_i)} \end{bmatrix} \tag{43}$$

where K_i is the number of breakpoints chosen to approximate the rolling moment due to δ_i, and the $\lambda_i^{(k)}$ are the normalized coefficients introduced earlier. The piecewise linear approximations for $M_i(\delta_i)$ and $N_i(\delta_i)$ follow accordingly. Furthermore, we have the following expression for δ_i given $\lambda_i^{(k)}$:

$$\delta_i = \sum_{k=1}^{K_i} \lambda_i^{(k)} \delta_i^{(k)} \tag{44}$$

We are now able to rewrite the $\boldsymbol{B}$ matrix as

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} L_1^{(1)} & L_1^{(2)} & \ldots & L_i^{(k)} & \ldots & L_m^{(K_m)} \\ M_1^{(1)} & M_1^{(2)} & \ldots & M_i^{(k)} & \ldots & M_m^{(K_m)} \\ N_1^{(1)} & N_1^{(2)} & \ldots & N_i^{(k)} & \ldots & N_m^{(K_m)} \end{bmatrix} \tag{45}$$

We also define a vector $\boldsymbol{\Lambda}$ as

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1^{(1)} \\ \lambda_1^{(2)} \\ \vdots \\ \lambda_i^{(k)} \\ \vdots \\ \lambda_m^{(K_m)} \end{bmatrix} \tag{46}$$

such that $\boldsymbol{B\delta}$ is replaced by $\tilde{\boldsymbol{B}}\boldsymbol{\Lambda}$. The vector $\boldsymbol{\Lambda}$ is of length

$$\sum_{i=1}^{m} K_i$$

and $\tilde{\boldsymbol{B}}$ is a matrix of size

$$3 \times \sum_{i=1}^{m} K_i$$

Note that if there are additional controlled variables, each one will add a row to $\tilde{\boldsymbol{B}}$. In formulating the piecewise linear control allocation problem, we no longer consider the case where the actuator rate limits and the sample rate of the flight control system set the upper and lower position limits on the control effectors. This is because we have made the assumption that the control moments are functions of control deflection only; therefore, we make no provision within the piecewise linear formulation of the problem for including actuator rates. If so desired, one could impose constraints on the $\lambda_i^{(k)}$ that limit how much it can change in one time step, thereby imposing a rate limit. In addition, the upper and lower position limits ($\boldsymbol{\delta}_{\max}$ and $\boldsymbol{\delta}_{\min}$) for each effector are now accounted for in the a priori selection of each control effector's breakpoints and are, therefore, automatically included in the problem formulation. Because we have replaced an explicit dependence on δ_i with an implicit one, $\lambda_i^{(k)}$, we only need to impose the following bounds: $\lambda_i^{(k)} \geq 0, k = 1, \ldots, K_i, i = 1, \ldots, m$. Recall that we do not need to define the upper bounds on the $\lambda_i^{(k)}$ explicitly because we have the constraint that

$$\sum_{k=1}^{K_i} \lambda_i^{(k)} = 1, \qquad i = 1, \ldots, m$$

and the constraints associated with the binary decision variables $y_i^{(k)}$ that will restrict $\lambda_i^{(k)} \leq 1$. Once we obtain an optimal solution to the problem, we compute each δ_i using Eq. (44). There are also an additional m constraints of the form

$$\sum_{k=1}^{K_i} \lambda_i^{(k)} = 1, \qquad i = 1, \ldots, m \tag{47}$$

Transformation to the MILP form requires additional constraints involving the binary variables $y_i^{(k)}$. These constraints are necessary to enforce the adjacency constraint given by Eq. (15).

$$\lambda_i^{(1)} \leq y_i^{(1)}, \qquad i = 1, \ldots, m \tag{48}$$

$$\lambda_i^{(k)} \leq y_i^{(k-1)} + y_i^{(k)}, \qquad i = 1, \ldots, m, \qquad k = 2, \ldots, K_i - 1 \tag{49}$$

$$\lambda_i^{K_i} \le y_i^{(K_i-1)}, \qquad i = 1, \ldots, m \tag{50}$$

$$\sum_{k=1}^{K_i-1} y_i^{(K_i-1)} = 1, \qquad i = 1, \ldots, m \tag{51}$$

$$y_i^{(k)} \in \{0, 1\} \tag{52}$$

The control deficiency branch in the form of a MILP is

$$\min_{\delta_s} J_D = \begin{bmatrix} 0 & \cdots & 0 \,\vdots\, 0 & \cdots & 0 \,\vdots\, \boldsymbol{w}_a^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} \\ \hdashline \boldsymbol{y} \\ \hdashline \boldsymbol{\delta}_s \end{bmatrix} \tag{53}$$

subject to

$$\boldsymbol{\delta}_s \ge \boldsymbol{0} \tag{54}$$

$$\tilde{\boldsymbol{B}}\boldsymbol{\Lambda} + \boldsymbol{\delta}_s \ge \boldsymbol{d}_{\text{des}} \tag{55}$$

$$-\tilde{\boldsymbol{B}}\boldsymbol{\Lambda} + \boldsymbol{\delta}_s \ge -\boldsymbol{d}_{\text{des}} \tag{56}$$

$$\lambda_i^{(k)} \ge 0, \qquad i = 1, \ldots, m, \qquad k = 2, \ldots, K_i \tag{57}$$

$$\sum_{k=1}^{K_i} \lambda_i^{(k)} = 1, \qquad i = 1, \ldots, m \tag{58}$$

$$\lambda_i^{(1)} \le y_i^{(1)}, \qquad i = 1, \ldots, m \tag{59}$$

$$\lambda_i^{(k)} \le y_i^{(k-1)} + y_i^{(k)}, \qquad i = 1, \ldots, m, \quad k = 2, \ldots, K_i - 1 \tag{60}$$

$$\lambda_i^{K_i} \le y_i^{(K_i-1)}, \qquad i = 1, \ldots, m \tag{61}$$

$$\sum_{k=1}^{K_i-1} y_i^{(K_i-1)} = 1, \qquad i = 1, \ldots, m \tag{62}$$

$$y_i^{(k)} \in \{0, 1\} \tag{63}$$

Note that the vector $\boldsymbol{y}$ is of length

$$-m + \sum_{i=1}^{m} K_i$$

and is defined in the same manner as $\boldsymbol{\Lambda}$:

$$\boldsymbol{y} = \begin{bmatrix} y_1^{(1)} & \ldots & y_1^{(K_1)} & \ldots & y_i^{(k)} & \ldots & y_m^{(K_m)} \end{bmatrix}^T \tag{64}$$

Control Sufficiency Branch

If there is sufficient control power available such that $J_D = 0$, then there may be excess control power available to optimize a subobjective. The subobjective could involve driving the control effectors to a preferred position $\boldsymbol{\delta}_p$. An optimization problem that reflects this objective is given by

$$\min_{\delta_s} J_S = \left\| \boldsymbol{w}_\delta^T (\boldsymbol{\delta} - \boldsymbol{\delta}_p) \right\|_1 \tag{65}$$

subject to

$$\boldsymbol{B}\boldsymbol{\delta} = \boldsymbol{d}_{\text{des}} \tag{66}$$

$$\underline{\boldsymbol{\delta}} \le \boldsymbol{\delta} \le \bar{\boldsymbol{\delta}} \tag{67}$$

where $\boldsymbol{w}_\delta^T$ is a column vector that allows one to weight one effector over another. This optimization problem can be cast into the linear programming (LP) framework, as follows:

$$\min_{\delta_s} J_S = \boldsymbol{w}_\delta^T \boldsymbol{\delta}_s \tag{68}$$

subject to

$$\begin{bmatrix} \boldsymbol{\delta}_s \\ -\boldsymbol{\delta} \\ \boldsymbol{\delta} \\ -\boldsymbol{\delta} + \boldsymbol{\delta}_s \\ \boldsymbol{\delta} + \boldsymbol{\delta}_s \end{bmatrix} \ge \begin{bmatrix} \boldsymbol{0} \\ -\bar{\boldsymbol{\delta}} \\ \underline{\boldsymbol{\delta}} \\ -\boldsymbol{\delta}_p \\ \boldsymbol{\delta}_p \end{bmatrix} \tag{69}$$

$$\boldsymbol{B}\boldsymbol{\delta} = \boldsymbol{d}_{\text{des}} \tag{70}$$

where $\boldsymbol{\delta}_s$, $\boldsymbol{\delta}_p$, and $\boldsymbol{w}_\delta$ are of the same dimension as the number of control effectors. The preference vector $\boldsymbol{\delta}_p$ is used to drive the effectors to some desired position.

Control Sufficiency Branch as a MILP

The differences between the MILP formulation of the control deficiency branch and MILP formulation of the control sufficiency branch are relatively minor. The primary difference is that the control sufficiency branch is a slightly larger problem because the objective function is being optimized with respect to the control effectors rather than the moments. This results in additional constraints due to the presence of two inequality constraints of the form $\boldsymbol{\delta} + \boldsymbol{\delta}_s \ge \boldsymbol{\delta}_p$ in the linear program. The number of binary variables $y_i^{(k)}$ and the parameters $\lambda_i^{(k)}$ are the same as for the control deficiency branch.

The control sufficiency branch can be stated as the following MILP:

$$\min_{\delta_s} J_S = \begin{bmatrix} 0 & \cdots & 0 \,\vdots\, 0 & \cdots & 0 \,\vdots\, \boldsymbol{w}_\delta^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} \\ \hdashline \boldsymbol{y} \\ \hdashline \boldsymbol{\delta}_s \end{bmatrix} \tag{71}$$

subject to

$$\boldsymbol{\delta}_s \ge \boldsymbol{0} \tag{72}$$

$$\sum_{k=1}^{K_i} \lambda_i^{(k)} \delta_i^{(k)} + \delta_{s,i} \ge \delta_{p,i}, \qquad i = 1, \ldots, m \tag{73}$$

$$-\sum_{k=1}^{K_i} \lambda_i^{(k)} \delta_i^{(k)} + \delta_{s,i} \ge -\delta_{p,i}, \qquad i = 1, \ldots, m \tag{74}$$

$$\tilde{\boldsymbol{B}}\boldsymbol{\Lambda} = \boldsymbol{d}_{\text{des}} \tag{75}$$

$$\lambda_i^{(k)} \ge 0, \qquad i = 1, \ldots, m, \qquad k = 2, \ldots, K_i \tag{76}$$

$$\sum_{k=1}^{K_i} \lambda_i^{(k)} = 1, \qquad i = 1, \ldots, m \tag{77}$$

$$\lambda_i^{(1)} \le y_i^{(1)}, \qquad i = 1, \ldots, m \tag{78}$$

$$\lambda_i^{(k)} \le y_i^{(k-1)} + y_i^{(k)}, \qquad i = 1, \ldots, m, \qquad k = 2, \ldots, K_i - 1 \tag{79}$$

$$\lambda_i^{K_i} \le y_i^{(K_i-1)}, \qquad i = 1, \ldots, m \tag{80}$$

$$\sum_{k=1}^{K_i-1} y_i^{(K_i-1)} = 1, \qquad i = 1, \ldots, m \tag{81}$$

$$y_i^{(k)} \in \{0, 1\} \tag{82}$$

Uniqueness of Solutions to the Nonlinear Control Allocation Problem

In the preceding discussion, the nonlinear control allocation problem was posed as a two-branch optimization problem where each optimization problem can be considered as an approximation to a nonlinear programming problem. Because we are now solving approximate nonlinear programming problems, we must address the issue of the uniqueness of the solutions to the control allocation problem. In the nonlinear control allocation problem, the control moments are often nonmonotonic functions of the control effectors; therefore, the solution to an instance of the control allocation problem is not guaranteed to be unique.

Although uniqueness is an issue in both the control deficiency and control sufficiency branches, we will concentrate on the uniqueness of the control sufficiency branch. Recall that when the control deficiency branch indicates a feasible solution to the control allocation problem exists, we solve a second optimization problem to minimize the control deflections relative to some preferred position. Now that we are solving an optimization problem that uses piecewise linear functions, the uniqueness of the sufficiency branch is dependent on the selection of the preference vector $\boldsymbol{\delta}_p$. Hereafter, we give conditions for which a finite number of unique solutions to the unweighted control sufficiency branch exist. We then give a method for testing the uniqueness of a given solution to $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ when $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$ is piecewise linear.

Conjecture: A finite number of solutions to the problem

$$\min J_s = \|\boldsymbol{\delta} - \boldsymbol{\delta}_p\|_1 \tag{83}$$

subject to

$$\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}} \tag{84}$$

$$\boldsymbol{\delta}_{\min} \leq \boldsymbol{\delta} \leq \boldsymbol{\delta} \tag{85}$$

where $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$ is piecewise linear, exist if the following condition is satisfied:

Condition: No facet of the hypersurface defined by the rows of $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ has a gradient equal to $\alpha[\pm 1 \ \ \pm 1 \ \ \ldots \ \ \pm 1] \, \forall \alpha \in \mathbb{R}$.

Geometrically, a row of $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ defines a faceted hypersurface. If any facet of $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ is coincident with a hyperplane on the boundary of a hypercube that is centered at $\boldsymbol{\delta}_p$, and whose vertices lie one-line segments parallel to the coordinate axes in $\mathbb{R}^n$, then an infinite number of solutions can exist. This is because any solution on this facet will have be the same distance from $\boldsymbol{\delta}_p$ in a one-norm sense.

Remark 1: If the condition is not satisfied, then a finite number of solutions will exist if $\boldsymbol{\delta}_p$ lies outside of any hypercube whose bisecting diagonal hyperplane is coincident with any of the hyperplanes defined by any row of $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ whose gradient is equal to $\alpha[\pm 1 \ \ \pm 1 \ \ \ldots \ \ \pm 1]$.

Remark 2: If the condition is satisfied, then no more than 2^n solutions exist, where n is the dimension of the control space, that is, $\boldsymbol{\delta} \in \mathbb{R}^n$. All candidate solutions lie on the vertices of a hypercube with sides of length $J_s\sqrt{n}$ where $J_s = \|\boldsymbol{\delta}^* - \boldsymbol{\delta}_p\|_1$ and $\boldsymbol{\delta}^* = \arg\min \|\boldsymbol{\delta} - \boldsymbol{\delta}_p\|_1$ subject to $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$.

Uniqueness Test: If the condition is satisfied, then the uniqueness of any solution to the control sufficiency branch can be tested in the following manner: Given a solution $\boldsymbol{\delta}^*$ that minimizes $\|\boldsymbol{\delta} - \boldsymbol{\delta}_p\|_1$ subject to $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$, and a preference vector $\boldsymbol{\delta}_p$, let $J_s = \|\boldsymbol{\delta}^* - \boldsymbol{\delta}_p\|_1$. If $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) = \boldsymbol{d}_{\mathrm{des}}$ at any of the 2^n vertices of the hypercube centered at $\boldsymbol{\delta}_p$ with sides of length $J_s\sqrt{n}$, then the solution is not unique. The vertices of the hypercube are given by $\boldsymbol{\delta} = \boldsymbol{\delta}_p \pm J_s \boldsymbol{e}_i$, where $\boldsymbol{e}_i^T = [0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]$ defines a unit vector parallel to the ith coordinate axis.

We show in Fig. 3 a set of pitching moment contours for a reentry vehicle. This set of contours is the result of the deflection of two control effectors, and of consideration of all possible combinations of (δ_1, δ_2). Note that, for this particular $\boldsymbol{\delta}_p$, there are two solutions to the sufficiency branch. Given one solution to the control allocation problem that minimizes $\|\boldsymbol{\delta} - \boldsymbol{\delta}_p\|_1$ for $\boldsymbol{d}_{\mathrm{des}} = 0.01$, one can check each vertex of the square centered on $\boldsymbol{\delta}_p$ and discover that there is a second solution to the optimization problem. If any of the line segments that describe the contours of constant moment have a slope of ± 1 and are coincident with any side of any square with a center at $\boldsymbol{\delta}_p$, then the uniqueness test would, at most, find two solutions, despite that an infinite number of solutions exist along the line segment. Note from Fig. 3 that there may be

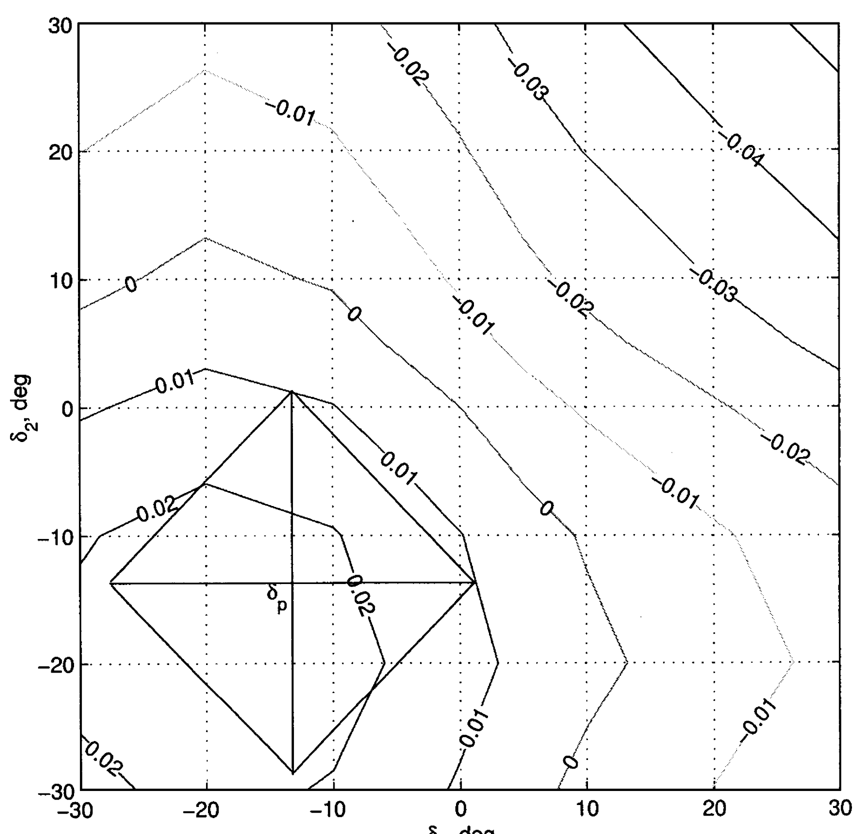


Fig. 3 Example δ_p, for which there exist two solutions to the sufficiency branch.

multiple preference vectors for which there are multiple solutions for a desired value of the control moment. Therefore, it is important that the uniqueness test be used in conjunction with the condition when testing for uniqueness. Uniqueness can be ensured by proper selection of $\boldsymbol{\delta}_p$ and/or the use of a weighted one-norm objective function where the weights are selected to enforce uniqueness. Although the foregoing remarks pertain specifically to the unweighted one-norm case, similar arguments can be developed for the weighted case.

Applications

There are two important applications for control allocation algorithms that take into account the nonlinear relationship between the control effectors and the control moments. Motivation for the use of nonlinear control allocation in the context of adaptive/reconfigurable control systems was outlined in an earlier section. In this section, we will compare the performance of the piecewise linear control allocation approach to a linear allocation method in a simulation of a reentry vehicle that uses a dynamic inversion control law for inner-loop control. A second application of nonlinear control allocation is the determination of constraints for use with trajectory reshaping algorithms.[17] Here, it is necessary to be able to determine accurately the range of angles of attack over which the vehicle can be trimmed in the presence of control effector failures. An additional benefit of solution of the constraint estimation problem is that trim lift and drag effects can be included in the trajectory reshaping problem with little additional computational burden. Both of these subproblems require that a control allocation problem be solved.

Simulation of Dynamic Inversion Control Law

The MILPs for the multibranch control allocation discussed earlier were implemented in a Simulink® simulation of a reentry vehicle. This particular vehicle has six control surfaces: left and right rudders, left and right flaperons, a body flap, and a speed brake. The simulation models the descent, final approach, and touchdown of the vehicle.

The performance of the piecewise linear approach is compared to that of a linear control allocation method. The linear control allocator assumes that the moments are linear functions of the effectors. The slope of the control moment curve is calculated with respect to the current control effector position by the use of a forward difference approximation. An intercept correction[12] term is then applied to account for mild nonlinearities in the aerodynamic data.

The results that follow give the closed-loop vehicle performance when there are two failures injected into the flight control system at different times during the approach and landing phases. It is assumed that there is some type of fault detection capability on-board the aircraft to identify the failures. The failure information is immediately passed to the control allocation algorithm to facilitate reconfiguration of the vehicle's effectors. The aircraft's trajectory begins at an altitude of about 15,000 ft above the runway and 4 miles downrange from the runway threshold. The first failure occurs 30 s into the simulation and involves the body flap being locked at −5 deg. This failure contributes a constant pitching moment to the aircraft. A second failure, where the right rudder becomes locked at 1 deg, occurs at 40 s. This particular failure adds not only a pitching moment to the aircraft, but also rolling and yawing moments. This particular failure combination was chosen because it requires the flaperons to operate in a highly nonlinear region of the control moment curve. After the failures are introduced, the aircraft tries to follow the nominal approach trajectory to the runway threshold. The aircraft extends the landing gear at about 68 s and flares immediately before touchdown. The simulation ends at touchdown when the weight-on-wheels switch is triggered.

For the control sufficiency branch of the control allocator, a preferred control position $\boldsymbol{\delta}_p$ is required. There are several different objectives that may be used to determine $\boldsymbol{\delta}_p$. These include, but are not limited to, minimum control deflection ($\boldsymbol{\delta}_p = \mathbf{0}$), minimum two-norm deflection, and null-space injection.[5,18] The preference vector used in this simulation is the minimum two-norm of the control surface deflection. This particular $\boldsymbol{\delta}_p$ minimizes $\boldsymbol{\delta}^T \boldsymbol{W} \boldsymbol{\delta}$ subject to $\boldsymbol{B}\boldsymbol{\delta} = \boldsymbol{d}_{\text{des}}$, where $\boldsymbol{W}$ is a positive definite weighting matrix. For our results, we take the weighting matrix $\boldsymbol{W}$ as the identity matrix. The corresponding solution to this problem is then $\boldsymbol{\delta}_p = \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-1}\boldsymbol{d}_{\text{des}}$. Note that this particular preference vector has the advantage of facilitating robustness analysis with the control allocator in the loop because the control allocator can be represented in closed-form for local linear analysis. For the piecewise linear control allocation, we found that it was sufficient to compute the right pseudo-inverse solution $\boldsymbol{\delta}_p$ with a $\boldsymbol{B}$ matrix that uses local slopes of the control moments at the last control surface position.

We will measure the performance of the two control allocators by their ability to produce deflections that, when applied to the nonlinear aerodynamic database, produce the desired moments about each axis. This metric is an indication of the error that results from the selection of a particular model in the control allocation algorithm.

Simulation Results

The results for the piecewise linear control allocator as compared to a linear control allocator with intercept correction are given in Figs. 4 and 5. Figure 4 shows the base 10 logarithm of $\|\boldsymbol{d}_{\text{des}} - \boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})\|_2$, where $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$ is the moment that is applied to the vehicle when given the control deflections returned by the control allocator. It is evident that the piecewise linear control allocator returns control surface deflections that produce the desired moments. On the other hand, the linear control allocator has a significant error. Note that at the 40-s mark, when the second failure is introduced, both control allocators indicate that there is a moment deficiency due to control effector saturation. Beyond 60 s, the performance of the piecewise linear control allocator improves once the effectors are no longer saturated. The poor performance of the linear control allocator is primarily due to the modeling errors inherent in the linear approximation of the control moment curves. The control surface commands from each control allocator are shown in Fig. 5. We see that, after the second failure is injected into the simulation, the flaperon deflections saturate at their upper limit. Note that the flap and speedbrake commands for the piecewise linear allocator oscillate after about 65 s. This appears to be due to an oscillatory $\boldsymbol{d}_{\text{des}}$ and $\boldsymbol{\delta}_p$ that result from body-axis rate loop closures. For the linear control allocator, we see that the flaps, left rudder, and speedbrake exhibit large amplitude oscillations. Note that for this failure case, the vehicle that flew with the piecewise linear control allocator maintained controlled flight.

Note that both the linear and nonlinear control allocation approaches were solved with GNU Linear Programming Kit's linear programming simplex library for the former and the branch-and-bound solver for the latter. The simulation with the nonlinear control allocator and the branch-and-bound solver ran extremely slow when compared to the simulation with the linear program solver used for the linear control allocator. It was observed that it took approximately 8.5 times longer on average to find a solution to the MILP control allocation problem as compared to the LP approach. This computational burden may limit one's ability to utilize the MILP approach in a real-time, digital flight control system in the near future. Performance gains may be achieved by the solution of the piecewise linear control allocation problem via the simplex method with the restricted basis entry rules.[15]

Application to Trajectory Reshaping

The intent here is to show a brief, although important, application of nonlinear control allocation. One of the areas of active research regarding reusable launch vehicles is the online determination of new feasible trajectories following a control effector failure. It is desired to guide the aircraft safely from the time that the failure occurs to a safe abort and to recover the vehicle if possible. An important part of the determination of feasible trajectories is having accurate estimates of the ranges of angle of attack, for a given Mach number, at which the aircraft can be trimmed. Also, it is important to be able to estimate critical parameters, such as the maximum lift-to-drag ratio, that will impact the optimal trajectory.[17] It is not desirable to use a linear control allocation approach to determine whether the vehicle can be trimmed. The reason is that a linear control allocator

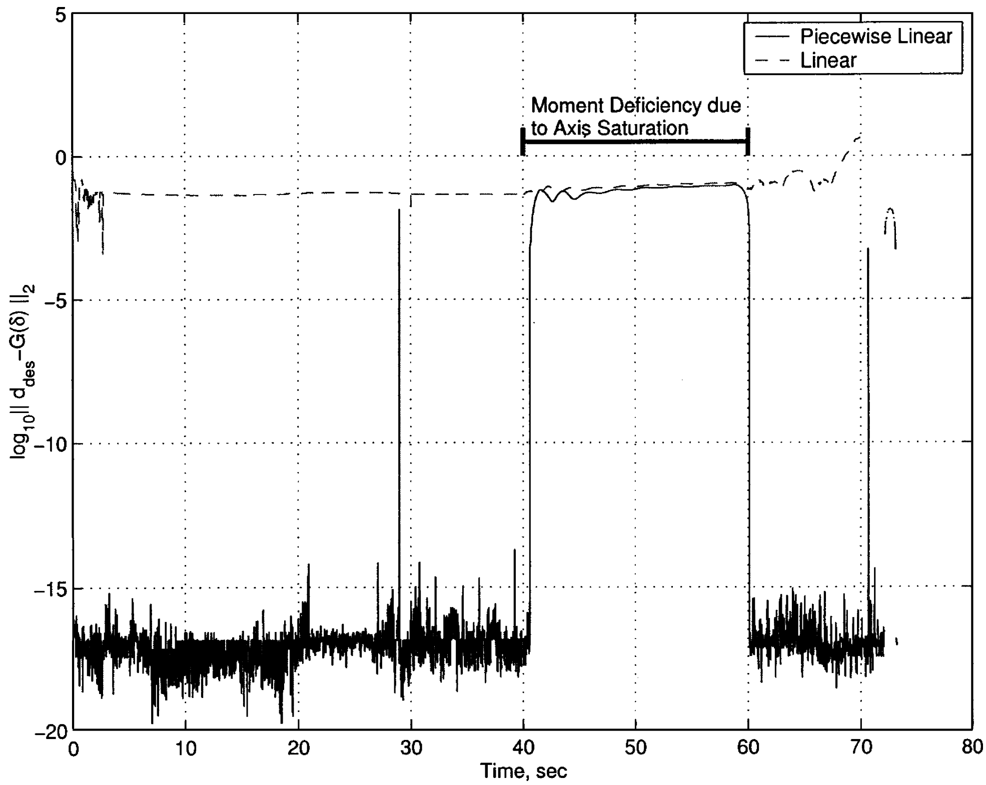


Fig. 4 Difference between desired and applied moments.

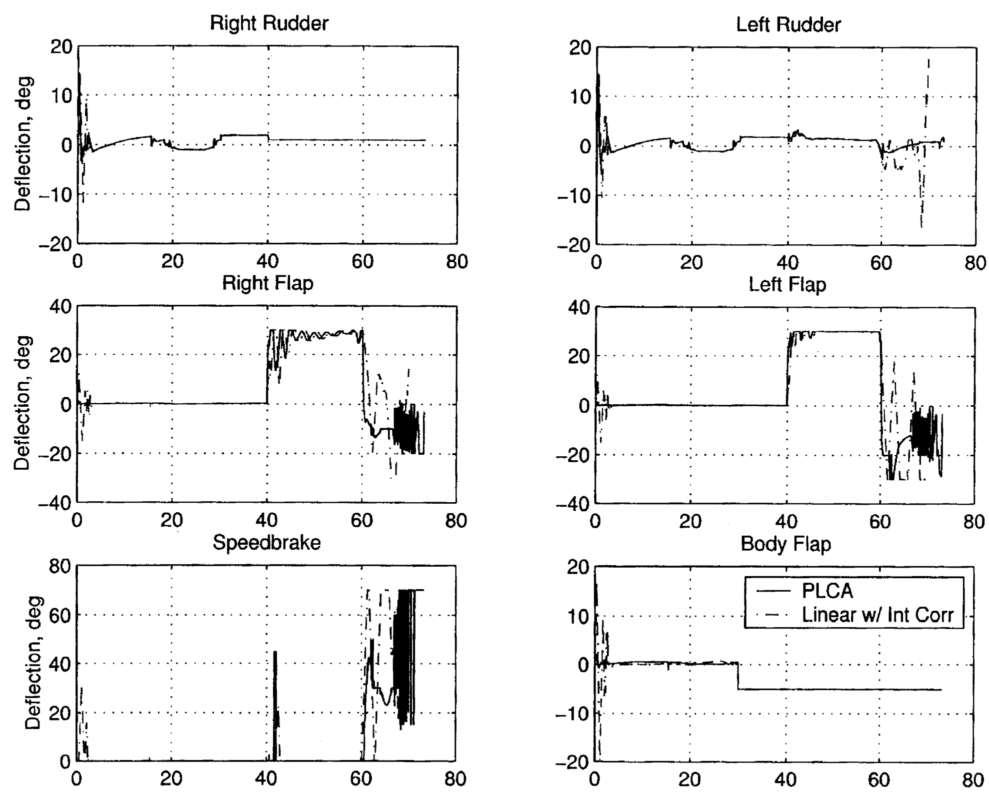


Fig. 5 Control effector time history.

would require a linear least-squares approximation of the nonlinear control moments, thereby introducing significant modeling errors. These modeling errors would then result in an inaccurate determination of the range of trimmable angles of attack. On the other hand, the use of a nonlinear control allocation algorithm allows one to estimate more accurately the trimmable angles of attack, as well as the critical parameters. Shown in Fig. 6 is an example of the moment deficiency for the same vehicle used in the aforementioned simulation when the right rudder is failed. The moment deficiency in this case is measured by the value of $\|\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta}) - \boldsymbol{d}_{\text{des}}\|_2$, where $\boldsymbol{G}(\boldsymbol{P}, \boldsymbol{\delta})$ is the piecewise linear representation of the vehicle aerodynamics and $\boldsymbol{d}_{\text{des}}$ are the base aerodynamic moments at each flight condition. A nonzero value of $\|\boldsymbol{G}(\boldsymbol{\delta}) - \boldsymbol{d}_{\text{des}}\|_2$ indicates that the vehicle cannot be trimmed at that particular Mach number and angle of attack. It is easily seen that a significant portion of the flight envelope is no longer flyable due to this effector failure. Figure 6 indicates that as

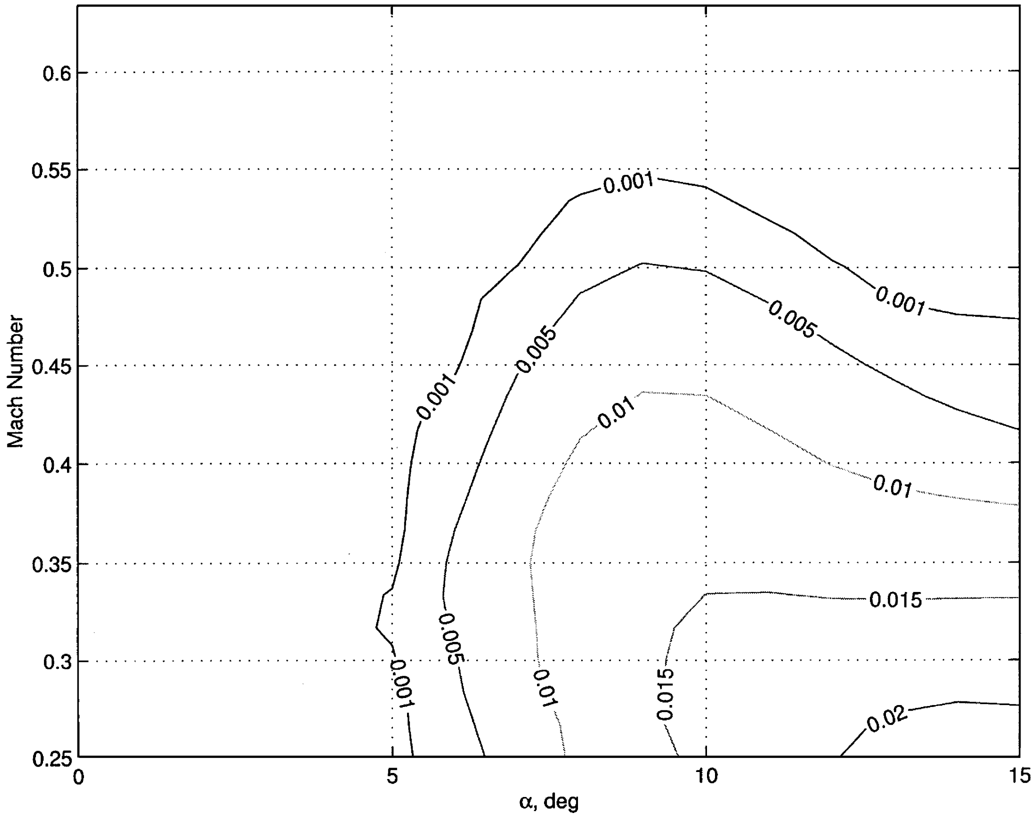


Fig. 6 Moment deficiency for a single effector failure.

the aircraft slows during approach and landing that the vehicle is unable to trim to angles of attack above 5 deg. Note that at Mach 0.5, the aircraft can be trimmed if the angle of attack is less than 7 deg or higher than 12.5 deg. Above Mach 0.55, the vehicle can be trimmed at any angle of attack for this particular failure.

Conclusions

A novel method was presented for the solution of a class of control allocation problems. Control allocation has historically been performed under the assumption that a linear relationship exists between the control induced moments and the control effector displacements. Because aerodynamic data almost always exhibit nonlinear behavior, such assumptions can lead to degraded performance or vehicle loss when secondary nonlinear effects must be used to control a vehicle, particularly after control effector failures have occurred. Aerodynamic databases are usually discrete valued and are almost always stored in multidimensional lookup tables where it is assumed that the data are connected by piecewise linear functions.

The approach that was presented assumes that the control effector moment data are piecewise linear. This assumption allows us to cast the control allocation problem as a piecewise linear program. To solve the piecewise linear program, it was reformulated as a mixed-integer linear program and solved with a readily available branch-and-bound algorithm. Simulation showed that the piecewise LP formulation results in improved tracking performance of the desired moments when compared to a more traditional control allocation approach that uses the linear assumption, especially when the aircraft is forced to operate with its control effectors in nonlinear portions of the control moment curves as a result of control effector failures. The piecewise linear control allocator was able to maintain control of the aircraft and land after the failures were introduced, whereas a control allocation algorithm that utilized a simple linear relationship along with an intercept correction term did not. The piecewise linear control allocator was applied to the determination of regions of trimmable angle of attack and Mach number for the purposes of trajectory reshaping under failure conditions. Note that linear control allocation is not suitable for this purpose.

Acknowledgments

This work was performed while the first author held a National Research Council Research Associateship Award at the U.S. Air Force Research Laboratory (AFRL). We thank B. R. Barmish of Case Western Reserve University for an insightful conversation regarding linear programming techniques for solving nonlinear root finding problems. We also acknowledge M. W. Oppenheimer, AFRL, for his assistance with the integration of the control allocator into the lifting-body simulation.

References

[1]Johnson, E., Calise, A., and Corban, J., "A Six Degree-of-Freedom Adaptive Flight Control Architecture for Trajectory Following," AIAA Paper 2002-4776, Aug. 2002.

[2]Wise, K., Brinker, J., Calise, A., Enns, D., Elgersma, M., and Voulgaris, P., "Direct Adaptive Reconfigurable Flight Control for a Tailless Advanced Fighter Aircraft," *International Journal of Robust and Nonlinear Control*, Vol. 9, No. 14, 1999, pp. 999–1012.

[3]Doman, D., and Ngo, A., "Dynamic Inversion-Based Adaptive/Reconfigurable Control of the X-33 on Ascent," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 2, 2002, pp. 275–284.

[4]Ngo, A., and Doman, D., "Footprint Determination for Reusable Launch Vehicles Experiencing Control Effector Failures," AIAA Paper 2002-4775, Aug. 2002.

[5]Buffington, J., "Modular Control Law Design for the Innovative Control Effectors (ICE) Tailless Fighter Aircraft Configuration 101-3," U.S. Air Force Research Lab., Tech. Rept. AFRL-VA-WP-TR 1999-3057, Wright–Patterson AFB, OH, June, 1999.

[6]Bodson, M., "Evaluation of Optimization Methods for Control Allocation," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 4, 2002, pp. 703–711.

[7]Page, A., and Steinberg, M. L., "A Closed-Loop Comparison of Control Allocation Methods," AIAA Paper 00-37180, Aug. 2000.

[8]Page, A. B., and Steinberg, M. L., "High-Fidelity Simulation Testing of Control Allocation Methods," AIAA Paper 2002-4547, May 2002.

[9]Durham, W., "Constrained Control Allocation," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 4, 1993, pp. 717–725.

[10]Durham, W., "Attainable Moments for the Constrained Control Allocation Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, 1994, pp. 1371–1373.

[11]Ikeda, Y., Hood, M., and Bolling, J., "Maximum Attainable Moment Space with L1 Optimization," AIAA Paper 01-37198, Aug. 2001.

[12]Doman, D., and Oppenheimer, M., "Improving Control Allocation Accuracy for Nonlinear Aircraft Dynamics," AIAA Paper 2002-4667, Aug. 2002.

[13]Doman, D., and Sparks, A., "Concepts for Constrained Control Allocation of Mixed Quadratic and Linear Effectors," American Automatic Control Council, Paper ACC02-AIAA1028, May 2002.

[14]Bolender, M., and Doman, D., "A Method for the Determination of the Attainable Moment Set for Non-Linear Control Effectors," *Proceedings of the 2003 IEEE Aerospace Conference* [CD ROM], Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 2003.

[15]Reklaitis, G., Ravindran, A., and Ragsdell, K., *Engineering Optimization: Methods and Application*, Wiley-Interscience, New York, 1983, pp. 314–324.

[16]Bertsimas, D., and Tsitsiklis, J., *Introduction to Linear Optimization*, Athena Scientific, Belmont, MA, 1997, pp. 15–18, 455–456.

[17]Schierman, J., Hull, J., and Ward, D., "On-Line Trajectory Command Reshaping for Reusable Launch Vehicles," AIAA Paper 2003-5439, Aug. 2003.

[18]Chandler, P., Pachter, M., and Mears, M., "System Identification for Adaptive and Reconfigurable Control," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 3, 1995, pp. 516–524.

The Fundamentals of Aircraft Combat Survivability: Analysis and Design, Second Edition

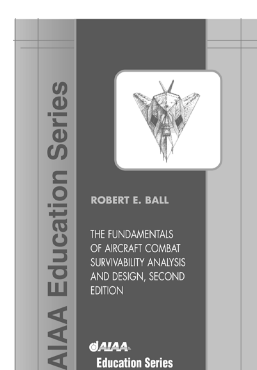
Robert E. Ball, Naval Postgraduate School

The extensively illustrated second edition of this best-selling textbook presents the fundamentals of the aircraft combat survivability design discipline as defined by the DoD military standards and acquisition processes. It provides the history of, the concepts for, the assessment methodology, and the design technology for combat survivability analysis and design of fixed- and rotary-wing aircraft, UAVs, and missiles. Each chapter specifies learning objectives; stresses important points; and includes notes, references, bibliography, and questions.

The Fundamentals of Aircraft Combat Survivability: Analysis and Design on CD-ROM is included with your purchase of the book. The CD-ROM gives you the portability and searchability that you need in your busy environment. A solutions manual is also available.

"The only book on the aircraft survivability discipline that speaks to both the operator and the engineer. THE bible of aircraft survivability!"

— Maj. Robert "Wanna" Mann
Chief, B-2 Branch
Wright-Patterson AFB



Contents:
- ▼ **An Introduction to the Aircraft Combat Survivability Discipline**
- ▼ **Aircraft Anatomy**
- ▼ **The Missions, the Threats and the Threat Effects**
- ▼ **Susceptibility (Ph and Pf)**
- ▼ **Vulnerability (Pk/h and Pk/f)**
- ▼ **Survivability (Ps and Pk)**
- ▼ **Appendices**

AIAA
American Institute of Aeronautics an
Publications Customer Service, P.O. Box 960, Herndon, VA 20172-0960
Fax: 703/661-1501 • Phone: 800/682-2422; 703/661-1595 • E-mail: warehouse@aiaa.org
Order 24 hours a day at: www.aiaa.org

AIAA Education Series
2003 • 950 pp • Mixed media • 1-56347-582-0
List Price: $105.95 • **AIAA Member Price: $69.95**

03-0608